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Asymptotic behavior of Non-Autonomous Navier-Stokes Type Equations

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ABSTRACT
In this paper we study the Cauchy-Dirichlet problem for a modified non-autonomous modified Navier-Stokes equation in a bounded domain. The existence and uniqueness of a weak solution of the problem are proved by Galerkin method. We then show the existence of a unique minimal $\mathcal{D}$-pullback attractor for the process associated to the problem with respect to a large class of non-autonomous forcing terms. Finally, when the force is time-independent and "small", the existence, uniqueness and stability of a stationary solution are also investigated.

KEYWORDS
modified Navier-Stokes equations; weak solution; stationary solution; pullback attractor; stability.

1. Introduction

One of the fundamental open problems for the Navier-Stokes equations in three-dimensional space is that of the uniqueness of the solution of the Cauchy-Dirichlet problem, which is not guaranteed in the functional classes in which global existence holds. It is therefore natural to investigate whether it is possible to modify the classical Navier-Stokes equations in a physically meaningful way, so as to obtain some new equations for which the Cauchy-Dirichlet problem is globally well-posed.

As we know, derivation of the classical Navier-Stokes equations from the general equations of conservation of momentum is based on the assumption that the component τ_{ij} of the stress tensor $T = \{\tau_{ij}\}$ depends linearly on the component $\frac{1}{2}\{\partial_i u_j + \partial_j u_i\}$ of the deformation velocity tensor D, namely,

$$\tau_{ij} = -p\delta_{ij} + \frac{\mu}{2}\left(\partial_i u_j + \partial_j u_i\right),$$

where p denotes the pressure of the fluid, μ is a positive constant representing the coefficient of kinematic viscosity.

A first interesting modification of this relationship is due to Ladyzhenskaya [7] who supposes T to be a continuous function of the components of D satisfying some further conditions and losing its linear feature for large value of the gradient of the velocity. The existence and long-time behaviour of solutions to this kind of modified Navier-Stokes equations have been studied extensively in [6–9, 17].

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In the work [13], Prouse remarks that for every high velocity and turbulent flow, there is no experimental evidence that the linear relationship between the stress tensor T and the deformation velocity tensor D continues to hold; he assumes that it holds when the velocity of the fluid is small and that it changes in a physically significant way otherwise. More precisely, he assumes that the relationship between the stress tensor T and the deformation velocity tensor D is given by

$$\tau_{ij} = -p\delta_{ij} + \frac{1}{2}\Big(\partial_i\varphi_j(u) + \partial_j\varphi_i(u)\Big),$$

where $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is function satisfying certain conditions below. If we introduce this relationship into a general equations of the conservation of momentum, we get another modified Navier-Stokes equations for incompressible fluid. The existence and long-time behaviour of solutions of these modified Navier-Stokes equations have been studied in [4, 5, 13, 14].

In this paper we consider the modified Navier-Stokes equations, proposed by Prouse [13], for incompressible fluids subject to a non-autonomous external force f in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$ of class $C^{1,1}$,

$$\left.\begin{array}{c} \partial_t u - \Delta\varphi(u) + (u\cdot\nabla)u + \nabla p - \nabla(\nabla\cdot\varphi(u)) = f,\ x \in \Omega, t > \tau, \\ \nabla\cdot u = 0 \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [\tau, +\infty) \\ u(x,\tau) = u_0(x) \quad x \in \Omega, \end{array}\right\} \tag{1}$$

where $\tau \in \mathbb{R}$; $u = u(x,t) = (u_1, \ldots, u_N)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, u_0 the initial velocity, other symbols satisfy the following conditions:

(H1) $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $\varphi(u) = \sigma(|u|)u$, with

$$\left\{\begin{array}{l} \sigma \in C^1([0, +\infty)), \\ \sigma(\xi) \geq \nu > 0 \text{ and } \sigma'(\xi) \geq 0 \text{ for all } \xi \geq 0, \\ \alpha\xi^{s-1} \leq \sigma(\xi) \leq \beta\xi^{s-1} \text{ for all } \xi \geq \xi_0, \end{array}\right. \tag{2}$$

where α, β, ξ_0, s are positive constants and $s \geq 1$ (if $N = 2$) or $s \geq N + 1$ (if $N \geq 3$);

(H2) $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies that

$$\int_{-\infty}^0 e^{\nu\lambda_1 s}\|f(s)\|^2_{V'}ds < +\infty,$$

where V' is the dual of the space V defined in Section 2, $\lambda_1 > 0$ is the first eigenvalue of operator $A := -\Delta_D$ in Ω with the homogeneous Dirichlet condition.

The aim of this paper is to continue studying the long-time behaviour of weak solutions to problem (1) under a more general class of time-dependent external forces. We first prove the existence and uniqueness of weak solutions by using the Galerkin method. To study the long-time behaviour of the solutions, we will use the theory of $\mathcal{D}$-pullback attractors that has been developed recently and has shown to be very useful in the understanding of the dynamics of non-autonomous dynamical system because it allows us to consider a larger class of non-autonomous forces than the theory of uniform attractors does (see [1]). We will show the existence of a $\mathcal{D}$-pullback attractor for the process associated to the problem. Furthermore, when the force is time-independent and "small", we prove the existence, uniqueness and stability of a stationary solution. The obtained results, in particular, extend and recover some known results in [4] and results for 2D classical Navier-Stokes equations in bounded domains [1, 15, 16] by taking $N = 2$, $s = 1$, $\alpha = \nu$, $\xi_0 = 0$.

Compared with the 2D Navier-Stokes equations, we here need treat the nonlinear term $-\Delta\varphi(u)$, which is not monotone and makes the equation fully nonlinear. Moreover, since we have no information about the sign of $\langle\Delta\varphi(u), \Delta u\rangle$, to obtain estimates, we must choose as test function $\Delta^{-1}u$ and this leads to a loss of regularity of solutions. This brings some additional difficulty in studying equation (1).

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of weak solutions to problem (1). In Section 3, we prove the existence of the $\mathcal{D}$-pullback attractor. In the last section, under some additional conditions, we prove the existence, uniqueness and stability of stationary solutions.

2. Existence and uniqueness of weak solutions

To set our problem in the abstract framework, we consider the following usual abstract spaces

$$\mathcal{V} = \{u \in [C_0^\infty(\Omega)]^N : \text{div } u = 0\}.$$

Let H be the closure of $\mathcal{V}$ in $\mathbb{L}^2$ with the norm $|.|$, and the inner product $(.,.)$, where for $u, v \in \mathbb{L}^2$,

$$(u, v) = \sum_{i=1}^{N} \int_\Omega u_i(x) v_i(x) dx,$$

V is the closure of $\mathcal{V}$ in $\mathbb{H}_0^1$ with norm $\|.\|$, and associated scalar product $((.))$, where for $u, v \in \mathbb{H}_0^1$,

$$((u, v)) = \sum_{i,j=1}^{N} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact.

Finally, we will use $\|.\|_*$ for the norm in V' and $\langle .,. \rangle$ for the duality pairing between V and V'.

Now we define the trilinear form b on V^3 by

$$b(u, v, w) = \sum_{i,j=1}^{N} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \text{for all } u, v, w \in V.$$

Set $A : V \to V'$ as $\langle Au, v \rangle = ((u, v))$, $B : V \times V \to V'$ as $\langle B(u, v), w \rangle = b(u, v, w)$. Denoting $D(A) = \mathbb{H}^2 \cap V$, then $Au = -P\Delta u$, for all $u \in D(A)$ (where P is the ortho-projector form $\mathbb{L}^2$ onto H).

Lemma 2.0.1. *[15] For all $u, v, w \in V$, we have*

$$b(u, v, w) = -b(u, w, v).$$

Hence

$$b(u, v, v) = 0.$$

Lemma 2.0.2. *[5, Lemma 3.1] For all $n \geq 2$, let $u, v \in \mathbb{L}^{n+2}$ and $w \in H$. Then for all $\mu > 0$ there exist constants $\Lambda_1(\mu) > 0$, $\Lambda_2(\mu) > 0$ such that*

$$|b(w, u, A^{-1}w)| \leq \mu |w|^2 + \Lambda_1(\mu) \|u\|_{\mathbb{L}^{n+2}}^{n+2} \|w\|_*^2,$$

and

$$|b(w, u, A^{-1}w) + b(v, w, A^{-1}w)| \leq \mu |w|^2 + \Lambda_2(\mu) \left(\|u\|_{\mathbb{L}^{n+2}}^{n+2} + \|v\|_{\mathbb{L}^{n+2}}^{n+2} \right) \|w\|_*^2.$$

Lemma 2.0.3. *For all $u, v \in \mathbb{L}^{s+1}$ and $w \in H$ with $s \geq 1$ (if $N = 2$) or $s \geq N+1$ (if $N \geq 3$). Then for all $\mu > 0$, there exists positive constants C_1, C_2 such that*

$$|b(w, u, A^{-1}w)| \leq \mu|w|^2 + C_1\mu^{-s}\|u\|_{\mathbb{L}^{s+1}}^{s+1}\|w\|_*^2, \tag{3}$$

and

$$|b(w, u, A^{-1}w) + b(v, w, A^{-1}w)| \leq \mu|w|^2 + C_2\mu^{-s}\left(\|u\|_{\mathbb{L}^{s+1}}^{s+1} + \|v\|_{\mathbb{L}^{s+1}}^{s+1}\right)|w\|_*^2. \tag{4}$$

Proof. In the case $N \geq 3$ or $N = 2$ and $s \geq 3$ then, (3) and (4) follow from Lemma 2.0.2 with $n = s - 1$.

In the case $N = 2$ and $s = 2$, we have

$$\begin{aligned} |b(w, u, A^{-1}w)| &\leq |w|\|u\|_{\mathbb{L}^3}\|A^{-1}w\|_{\mathbb{L}^6} \\ &\leq \frac{\mu}{2}|w|^2 + \frac{1}{2\mu}\|u\|_{\mathbb{L}^3}^2\|A^{-1}w\|_{\mathbb{L}^6}^2. \end{aligned}$$

Since $V \hookrightarrow \mathbb{L}^6$, then there exists $C_3 > 0$ such that

$$\|A^{-1}w\|_{\mathbb{L}^6}^2 \leq C_3\|A^{-1}w\|^2 = C_3\|w\|_*^2.$$

Using Young's inequality, we get

$$\|u\|_{\mathbb{L}^3}^2\|w\|_*^2 \leq \epsilon\|u\|_{\mathbb{L}^3}^3 + \frac{4}{27\epsilon^2}\|w\|_*^6 < \epsilon\|u\|_{\mathbb{L}^3}^3 + \epsilon^{-2}\|w\|_*^6.$$

Choose $\epsilon = \dfrac{\sqrt{C_3}\|w\|_*^2}{\mu\sqrt{\lambda_1}}$, we get

$$\begin{aligned} |b(w, u, A^{-1}w)| &\leq \frac{\mu}{2}|w|^2 + \frac{C_3}{2\mu}\|u\|_{\mathbb{L}^3}^2\|w\|_*^2 \\ &\leq \frac{\mu}{2}|w|^2 + \frac{C_3^{\frac{3}{2}}}{2\mu^2\lambda_1^{\frac{1}{2}}}\|u\|_{\mathbb{L}^3}^3\|w\|_*^2 + \frac{\mu\lambda_1}{2}\|w\|_*^2 \\ &\leq \mu|w|^2 + \frac{C_3^{\frac{3}{2}}}{2\lambda_1^{\frac{1}{2}}}\mu^{-2}\|u\|_{\mathbb{L}^3}^3\|w\|_*^2. \end{aligned}$$

□

Lemma 2.0.4. *[5, Lemma 3.2] Assume φ satisfies (2), then*

$$(A\varphi(v), v) \geq \nu\|v\|^2 \quad \text{for all } v \in V,$$

and

$$(\varphi(u) - \varphi(v), u - v) \geq \nu|u - v|^2 \text{ for all } u, v \in H.$$

Lemma 2.0.5. *Assume that φ satisfies (2) then for all $u \in \mathbb{L}^{s+1}$, we have*

$$(\varphi(u), u) \geq \alpha\|u\|_{\mathbb{L}^{s+1}}^{s+1} - \alpha\xi_0^{s+1}|\Omega|.$$

Proof. Putting

$$\Omega_1 = \{x \in \Omega : |u(x)| < \xi_0\} \text{ and } \Omega_2 = \{x \in \Omega : |u(x)| \geq \xi_0\}.$$

First, from (2) we have

$$\begin{aligned}
\int_\Omega |u(x)|^{s+1}dx &= \int_{\Omega_1} |u(x)|^{s+1}dx + \int_{\Omega_2} |u(x)|^{s-1}|u(x)|^2 dx \\
&\le \xi_0^{s+1}\int_{\Omega_1} dx + \frac{1}{\alpha}\int_{\Omega_2} \sigma(|u(x)|)|u(x)|^2 dx \\
&= \xi_0^{s+1}|\Omega_1| + \frac{1}{\alpha}\int_{\Omega_2} \sigma(|u(x)|)u(x).u(x)dx \\
&= \xi_0^{s+1}|\Omega_1| + \frac{1}{\alpha}\int_{\Omega_2} \varphi(u(x)).u(x)dx \\
&\le \xi_0^{s+1}|\Omega| + \frac{1}{\alpha}\int_{\Omega} \varphi(u(x)).u(x)dx \\
&= \xi_0^{s+1}|\Omega| + \frac{1}{\alpha}(\varphi(u), u).
\end{aligned}$$

i.e.,

$$(\varphi(u), u) \ge \alpha\|u\|_{\mathbb{L}^{s+1}}^{s+1} - \alpha\xi_0^{s+1}|\Omega|.$$

□

Lemma 2.0.6. *Let $u \in \mathbb{L}^r \cap V$ and $v \in V$. Then, for all $n \le r$, we have*

$$|b(u, u, v)| \le C_4\|u\|^{n/r}|u|^{1-n/r}\|v\|\|u\|_{\mathbb{L}^r}.$$

Proof. We have,

$$|b(u, v, w)| \le |uw|\|v\|.$$

By Hölder's inequality,

$$|uw| \le \|u\|_{\mathbb{L}^{n'}}^{n/r}|u|^{1-n/r}\|w\|_{\mathbb{L}^r}, \quad \left(\frac{1}{n'} = \frac{1}{2} - \frac{1}{n}\right).$$

By Sobolev's inequality, we have

$$\|u\|_{\mathbb{L}^{n'}} \le C_4\|u\|.$$

So, we get

$$|b(u, v, w)| \le C_4\|u\|^{n/r}|u|^{1-n/r}\|v\|\|w\|_{\mathbb{L}^r}.$$

Hence,

$$|b(u, u, v)| \le C_4\|u\|^{n/r}|u|^{1-n/r}\|v\|\|u\|_{\mathbb{L}^r}.$$

The lemma is proved. □

Definition 2.1. *A function u is called a weak solution of problem (1) on the interval (τ, T) if*

$$\begin{cases}
u \in L^2(\tau, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \cap L^{s+1}(\tau, T; \mathbb{L}^{s+1}), \\
\dfrac{d}{dt}u(t) + A\varphi(u(t)) + B(u(t), u(t)) = f(t) \text{ in } \mathcal{D}'(\tau, +\infty; V'), \\
u(\tau) = u_0.
\end{cases} \tag{5}$$

Theorem 2.1. *Assume that* $u_0 \in H$, φ *satisfies* $(H1)$ *and* f *satisfies* $(H2)$. *For any* $\tau \in \mathbb{R}, T > \tau$ *given, problem* (1) *has a unique weak solution* u *on* (τ, T). *Moreover, the following inequality holds,*

$$|u(t)|^2 \leq e^{-\nu\lambda_1(t-\tau)}|u_0|^2 + \frac{e^{-\nu\lambda_1 t}}{\nu\lambda_1}\int_{-\infty}^{t} e^{\nu\lambda_1 s}|f(s)|_*^2 ds. \tag{6}$$

Proof. *(i) Existence.* Let us consider $\{w_j\} \subset V$, the orthonormal basis of H of all the eigenfunctions of the Stokes problem in Ω with homogeneous Dirichlet conditions. The subspace of V spanned by $w_1, \cdots, w_m$ will be denoted V_m. Consider the projector $P_m : H \to V_m$ given by

$$P_m u = \sum_{i=1}^{m}(u, w_i)w_i,$$

and define

$$u^m(t) = \sum_{i=1}^{m}\gamma_{mi}(t)w_i,$$

where

$$\left.\begin{aligned} u^m \in L^2(\tau, T; V_m) \cap C^0([\tau, T]; V_m), \\ \frac{d}{dt}(u^m(t), w_i) + (A\varphi(u^m(t)), w_i) + b(u^m(t), u^m(t), w_i) = \langle f(t), w_i\rangle \\ \text{in } \mathcal{D}'(\tau, T),\ 1 \leq i \leq m, \\ u^m(\tau) = P_m u_0. \end{aligned}\right\} \tag{7}$$

Using the Peano theorem, we get the local existence of u^m on (τ, t^*). We now establish some *a priori* estimates for u^m. Because of following estimates, we can take $t^* = T$. We have

$$\frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + (A\varphi(u^m(t)), u^m(t)) + b(u^m(t), u^m(t), u^m(t)) = \langle f(t), u^m(t)\rangle.$$

Using Lemma 2.0.4, Lemma 2.0.1, and Cauchy's inequality, we get

$$\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 \leq \frac{1}{\nu}\|f(t)\|_*^2.$$

Integrating on $[\tau, t]$, $\tau \leq t \leq T$, we get

$$\begin{aligned} |u^m(t)|^2 + \nu\int_{\tau}^{t}\|u^m(s)\|^2 ds &\leq |u^m(\tau)|^2 + \frac{1}{\nu}\int_{\tau}^{t}\|f(s)\|_*^2 ds \\ &\leq |u_0|^2 + \frac{1}{\nu}\|f\|_{L^2(\tau,T;V')}^2. \end{aligned}$$

We get three constants (independent of m) C_5, C_6 and C_7 such that

$$\sup_{t\in[\tau,T]}|u^m(t)|^2 \leq C_5, \quad \int_{\tau}^{T}\|u^m(t)\|^2 dt \leq C_6, \quad \int_{\tau}^{T}\|u^m(t)\|_{\mathbb{L}^{s+1}}^{s+1} \leq C_7. \tag{8}$$

For the last estimate, we proceed as follows. In the case $N = 2$, inequality (8) follows from the facts that $V \hookrightarrow \mathbb{L}^{s+1}$. In the case $N > 2$, using Lemma 2.0.5, we have

$$(\varphi(u^m(t)), u^m(t)) \geq \alpha\|u^m(t)\|_{\mathbb{L}^{s+1}}^{s+1} - \alpha\xi_0^{s+1}|\Omega|.$$

Moreover, using Lemma 2.0.3 with $\mu = \delta$ ($\delta > 0$ will be chosen later), we get

$$\begin{aligned}|b(u^m, u^m, A^{-1}u^m)| &\le \delta|u^m|^2 + C_1\delta^{-s}\|u^m\|^{s+1}_{\mathbb{L}^{s+1}}\|u^m\|^2_* \\ &\le \delta|u^m|^2 + C_1\delta^{-s}\|u^m\|^{s+1}_{\mathbb{L}^{s+1}}\lambda_1^{-1}|u^m|^2 \\ &\le \delta C_5 + C_1\delta^{-s}\|u^m\|^{s+1}_{\mathbb{L}^{s+1}}\lambda_1^{-1}C_5.\end{aligned}$$

Now, we can choose $\delta > 0$ such that $C_1\delta^{-s}\lambda_1^{-1}C_5 = \dfrac{\alpha}{2}$ to obtain

$$2|b(u^m(t), u^m(t), A^{-1}u^m(t))| \le 2\delta C_5 + \alpha\|u^m(t)\|^{s+1}_{\mathbb{L}^{s+1}}.$$

From equation (7), we have

$$\frac{1}{2}\frac{d}{dt}\|u^m(t)\|^2_* + (\varphi(u^m(t)), u^m(t)) + b(u^m(t), u^m(t), A^{-1}u^m(t)) = \langle f(t), A^{-1}u^m(t)\rangle.$$

Integrating from τ to T, we get

$$\begin{aligned}&\|u^m(T)\|^2_* + 2\int_\tau^T (\varphi(u^m(t)), u^m(t))dt + 2\int_\tau^T b(u^m(t), u^m(t), A^{-1}u^m(t))dt \\ &\quad\le \|u^m(\tau)\|^2_* + \int_\tau^T \|f(t)\|^2_* dt + \int_\tau^T \|u^m(t)\|^2_* dt \\ &\quad\le \lambda_1^{-1}|u^m(\tau)|^2 + \|f\|^2_{L^2(\tau,T;V')} + \lambda_1^{-1}\int_\tau^T |u^m(t)|^2 dt \\ &\quad\le \lambda_1^{-1}|u_0|^2 + \|f\|^2_{L^2(\tau,T;V')} + \lambda_1^{-1}C_5(T-\tau).\end{aligned}$$

On the other hand,

$$\begin{aligned}&2\int_\tau^T (\varphi(u^m(t)), u^m(t))dt + 2\int_\tau^T b(u^m(t), u^m(t), A^{-1}u^m(t))dt \\ &\quad\ge 2\alpha\int_\tau^T \|u^m(t)\|^{s+1}_{\mathbb{L}^{s+1}}dt - 2\alpha\xi_0^{s+1}|\Omega|(T-\tau) - 2\delta C_5(T-\tau) - \alpha\int_\tau^T \|u^m(t)\|^{s+1}_{\mathbb{L}^{s+1}}dt \\ &\quad= \alpha\int_\tau^T \|u^m(t)\|^{s+1}_{\mathbb{L}^{s+1}}dt - \left(2\alpha\xi_0^{s+1}|\Omega| + 2\delta C_5\right)(T-\tau).\end{aligned}$$

Hence,

$$\alpha\int_\tau^T \|u^m(t)\|^{s+1}_{\mathbb{L}^{s+1}}dt \le \lambda_1^{-1}|u_0|^2 + \|f\|^2_{L^2(\tau,T;V')} + \left[2\alpha\xi_0^{s+1}|\Omega| + 2\delta C_5 + \lambda_1^{-1}C_5\right](T-\tau). \tag{9}$$

By condition (2), $\sigma \in C^1$ implies σ is bounded on $[0, \xi_0]$, i.e., $\sigma(\xi) \le C_\sigma$ for all $\xi \in [0, \xi_0]$. Thus,

$$\begin{aligned}\|\varphi(u)\|^{1+\frac{1}{s}}_{\mathbb{L}^{1+\frac{1}{s}}} &= \int_{\Omega_1} |\varphi(u(x))|^{1+\frac{1}{s}}dx + \int_{\Omega_2} |\varphi(u(x))|^{1+\frac{1}{s}}dx \\ &= \int_{\Omega_1} |\sigma(|u(x)|)|^{1+\frac{1}{s}}|u(x)|^{1+\frac{1}{s}}dx + \int_{\Omega_2} |\sigma(|u(x)|)|^{1+\frac{1}{s}}|u(x)|^{1+\frac{1}{s}}dx,\end{aligned}$$

where

$$\Omega_1 = \{x \in \Omega : |u(x)| < \xi_0\} \text{ and } \Omega_2 = \{x \in \Omega : |u(x)| \ge \xi_0\}.$$

Therefore,

$$\begin{aligned}\|\varphi(u)\|_{\mathbb{L}^{1+\frac{1}{s}}}^{1+\frac{1}{s}} &\leq \xi_0^{1+\frac{1}{s}} \int_{\Omega_1} |\sigma(|u(x)|)|^{1+\frac{1}{s}} dx + \int_{\Omega_2} \left(\beta|u(x)|^{s-1}\right)^{1+\frac{1}{s}} |u(x)|^{1+\frac{1}{s}} dx \\ &= \xi_0^{1+\frac{1}{s}} \int_{\Omega_1} |\sigma(|u(x)|)|^{1+\frac{1}{s}} dx + \int_{\Omega_2} (\beta|u(x)|^{s})^{1+\frac{1}{s}} dx \\ &\leq \xi_0^{1+\frac{1}{s}} \int_{\Omega_1} |\sigma(|u(x)|)|^{1+\frac{1}{s}} dx + \int_{\Omega_2} (\beta|u(x)|^{s})^{1+\frac{1}{s}} dx \\ &\leq \xi_0^{1+\frac{1}{s}} \int_{\Omega_1} C_\sigma^{1+\frac{1}{s}} dx + \beta^{1+\frac{1}{s}} \int_{\Omega_2} |u(x)|^{1+s} dx \\ &\leq \xi_0^{1+\frac{1}{s}} C_\sigma^{1+\frac{1}{s}} |\Omega| + \beta^{1+\frac{1}{s}} \|u\|_{\mathbb{L}^{1+s}}^{1+s}.\end{aligned}$$

So, we get

$$\int_\tau^T \|\varphi(u^m(s))\|_{\mathbb{L}^{1+\frac{1}{s}}}^{1+\frac{1}{s}} ds \leq \xi_0^{1+\frac{1}{s}} C_\sigma^{1+\frac{1}{s}} |\Omega|(T-\tau) + \beta^{1+\frac{1}{s}} C_7 = C_8. \tag{10}$$

Finally, since A is a continuous linear operator from $\mathbb{L}^{1+\frac{1}{s}} = \mathbb{W}^{0,1+\frac{1}{s}}$ into $\mathbb{W}^{-2,1+\frac{1}{s}}$, therefore

$$\int_\tau^T \|A\varphi(u^m(s))\|_{\mathbb{W}^{-2,1+\frac{1}{s}}}^{1+\frac{1}{s}} ds \leq C_9. \tag{11}$$

Then, $\{u^m\}$ is bounded in $L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \cap L^{s+1}(\tau, T; \mathbb{L}^{s+1})$.

Therefore, we have

$$\begin{array}{ll} u^m \to u & \text{weakly in } L^2(\tau, T; V), \\ u^m \to u & \text{weakly in } L^{s+1}(\tau, T; \mathbb{L}^{s+1}), \\ u^m \to u & \text{weakly star in } L^\infty(\tau, T; H), \\ \varphi(u^m) \to \chi & \text{weakly in } L^{1+\frac{1}{s}}(\tau, T; \mathbb{L}^{1+\frac{1}{s}}), \end{array}$$

up to a subsequence.

By rewriting the equation as

$$\frac{du^m(t)}{dt} = -A\varphi(u^m(t)) - B(u^m(t), u^m(t)) + f(t), \tag{12}$$

we see that $\left\{\frac{du^m}{dt}\right\}$ is bounded in $L^{1+\frac{1}{s}}(\tau, T; \mathbb{W}^{-2,1+\frac{1}{s}}) + L^{1+\frac{1}{s}}(\tau, T; V') + L^2(\tau, T; V')$, and therefore in $L^{1+\frac{1}{s}}(\tau, T; (\mathbb{W}_0^{2,s+1} \cap V)')$. Indeed, from (11), we have $A\varphi(u^m)$ is bounded in $L^{1+\frac{1}{s}}(\tau, T; \mathbb{W}^{-2,1+\frac{1}{s}})$.

In the case $N = 2$, using Lemma 2.0.6 with $r = N = 2$, we have

$$\begin{aligned}\int_\tau^T \|B(u^m(t), u^m(t))\|_*^2 dt &\leq C_4^2 \int_\tau^T \|u^m(t)\|^2 |u^m(t)|_{\mathbb{L}^2}^2 dt \\ &= C_4^2 \int_\tau^T \|u^m(t)\|^2 |u^m(t)|^2 dt \\ &\leq C_4^2 \|u^m\|_{L^\infty(\tau,T;H)}^2 \|u^m\|_{L^2(\tau,T;V)}^2 \\ &\leq C_4^2 C_5 C_6.\end{aligned}$$

In the case, $N > 2$, using Lemma 2.0.6 with $r = s + 1$, we have

$$\int_\tau^T \|B(u^m(t), u^m(t))\|_*^{1+\frac{1}{s}} dt \leq C_4^{1+\frac{1}{s}} \int_\tau^T \|u^m(t)\|^{\frac{N}{s}} |u^m(t)|^{\frac{s+1-N}{s}} \|u^m(t)\|_{\mathbb{L}^{s+1}}^{\frac{s+1}{s}} dt.$$

Since $s \geq N+1$, we have

$$\|u^m(t)\|^{\frac{2N}{s}} \leq 1 + \|u^m(t)\|^2,$$

and

$$\|u^m(t)\|_{\mathbb{L}^{s+1}}^{\frac{2(s+1)}{s}} \leq 1 + \|u^m(t)\|_{\mathbb{L}^{s+1}}^{s+1},$$

so we get

$$\begin{aligned}
&\int_\tau^T \|u^m(t)\|^{\frac{N}{s}} |u^m(t)|^{\frac{s+1-N}{s}} \|u^m(t)\|_{\mathbb{L}^{s+1}}^{\frac{s+1}{s}} dt \\
&\leq \|u^m\|_{L^\infty(\tau,T;H)}^{\frac{s+1-N}{s}} \left(\frac{1}{2}\int_\tau^T \|u^m(t)\|^{\frac{2N}{s}} dt + \frac{1}{2}\int_\tau^T \|u^m(t)\|_{\mathbb{L}^{s+1}}^{\frac{2(s+1)}{s}} dt\right) \\
&\leq C_5^{\frac{s+1-N}{s}} \left((T-\tau) + \frac{1}{2}\int_\tau^T \|u^m(t)\|^2 dt + \frac{1}{2}\int_\tau^T \|u^m(t)\|_{\mathbb{L}^{s+1}}^{s+1} dt\right) \\
&\leq C_5^{\frac{s+1-N}{s}} \left((T-\tau) + \frac{1}{2}C_6 + \frac{1}{2}C_7\right).
\end{aligned}$$

This implies that $B(u^m, u^m)$ is bounded in $L^{1+\frac{1}{s}}(\tau, T; V')$.

Notice that

$$V \subset\subset H \subset (\mathbb{W}_0^{2,s+1} \cap V)'$$

is an evolution triplet, applying the Aubin-Lions lemma [10] we can assume that $u^m \to u$ strongly in $L^2(\tau, T; H)$. Hence $u^m \to u$ a.e. in $\Omega \times [\tau, T]$. Since φ is continuous, it follows that $\varphi(u^m) \to \varphi(u)$ a.e. in $\Omega \times [\tau, T]$. Since the weak limit is unique and thanks to Lemma 1.3 in [10, Chapter 1], one has

$$\varphi(u^m) \to \varphi(u), \text{ in } L^{1+\frac{1}{s}}(\tau, T; \mathbb{L}^{1+\frac{1}{s}}).$$

Similarly,

$$B(u^m, u^m) \to B(u, u), \text{ in } L^{1+\frac{1}{s}}(\tau, T; V').$$

Now, we can take limits in (12) after integrating over the interval $(\tau; T)$, getting the result that u is a solution to our problem (5).

(ii) Uniqueness and continuous dependence. Assume that $u = u(t; \tau, u_0)$ and $v = v(t; \tau, v_0)$ be the weak solutions of (1). Set $w = u - v$, then

$$w \in L^2(\tau, T; V) \cap L^{N+2}(\tau, T; \mathbb{L}^{N+2})$$

and w satisfies

$$\frac{d}{dt}w + A(\varphi(u) - \varphi(v)) + B(u, u) - B(v, v) = 0$$

which we rewrite, using the bilinearity of B,

$$B(u, u) - B(v, v) = B(u - v, u) + B(v, u - v) = B(w, u) + B(v, w)$$

as

$$\frac{d}{dt}w + A(\varphi(u) - \varphi(v)) + B(w, u) + B(v, w) = 0.$$

Take inner by $A^{-1}w$ and using Lemma 2.0.4, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|_*^2 + \nu|w|^2 \le \nu|w|^2 + \Lambda_2(\nu)\left(\|u\|_{\mathbb{L}^{N+2}}^{N+2} + \|v\|_{\mathbb{L}^{N+2}}^{N+2}\right)\|w\|_*^2,$$

whence

$$\frac{d}{dt}\|w\|_*^2 \le K(t)\|w\|_*^2, \tag{13}$$

where $K(t) = 2\Lambda_2(\nu)\left(\|u(t)\|_{\mathbb{L}^{N+2}}^{N+2} + \|v(t)\|_{\mathbb{L}^{N+2}}^{N+2}\right) \in L^1(\tau, T)$, and

$$\|w(t)\|_*^2 \le \|w(\tau)\|_*^2 + \int_\tau^t K(s)\|w(s)\|_*^2 ds,$$

The Gronwall lemma implies now for any $t \ge \tau$

$$\|w(t)\|_*^2 \le \|u_0 - v_0\|_*^2 e^{\int_\tau^t K(s)ds}. \tag{14}$$

The last estimate implies that the solution is unique (if $u_0 = v_0$) and the continuous dependence of solution.

iii) The a priori estimate (6). Multiplying (5) by u and integrating over Ω, we have

$$\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \le \frac{1}{\nu}\|f(t)\|_*^2.$$

Noting that $\|u\|^2 \ge \lambda_1|u|^2$, we have

$$\frac{d}{dt}|u(t)|^2 + \nu\lambda_1|u(t)|^2 \le \frac{1}{\nu}|f(t)|_*^2.$$

Applying the Gronwall lemma we get (6). Hence it follows that the solution u can be extended to the whole interval $[\tau, +\infty)$. □

3. Existence of pullback attractors

For convenience of readers, we first recall some results on theory of pullback attractors which we will use in the paper.

A process on X is a two parameters process on X denoted by $U(t,\tau)$ which has the following properties

$$U(t,r)U(r,\tau) = U(t,\tau) \text{ for all } t \ge r \ge \tau,$$
$$U(\tau,\tau) = Id \text{ for all } \tau \in \mathbb{R}.$$

The process $\{U(t,\tau)\}$ is said to be norm-to-weak continuous if $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$, as $x_n \to x$ in X, for all $t \ge \tau$, $\tau \in \mathbb{R}$.

Suppose that $\mathcal{D}$ is a nonempty class of parameterized sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.

Definition 3.1. *The process $\{U(t,\tau)\}$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, and any sequence $\tau_n \to -\infty$, any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Definition 3.2. *The family of bounded sets $\hat{\mathcal{B}} \in \mathcal{D}$ is called pullback $\mathcal{D}$-absorbing for the process*

$U(t,\tau)$ if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \le t$ such that

$$\bigcup_{\tau \le \tau_0} U(t,\tau)D(\tau) \subset B(t).$$

Definition 3.3. *A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a pullback $\mathcal{D}$-attractor for $\{U(t,\tau)\}$ if*

(1) $A(t)$ is compact for all $t \in \mathbb{R}$;
(2) $\hat{\mathcal{A}}$ is invariant, i.e.,

$$U(t,\tau)A(\tau) = A(t), \text{ for all } t \ge \tau;$$

(3) $\hat{\mathcal{A}}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau \to -\infty} dist(U(t,\tau)D(\tau), A(t)) = 0, \text{ for all } \hat{\mathcal{D}} \in \mathcal{D}, \text{ and all } t \in \mathbb{R};$$

(4) If $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets then $A(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Theorem 3.1. *[11] Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, then $\{U(t,\tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau)B(\tau)}.$$

Due to the results of Theorem 2.1, we can define a two parameters process $U(t,\tau)$ in H by

$$U(t,\tau)u_0 = u(t;\tau,u_0), \qquad \tau \le t, u_0 \in H,$$

where $u(t) = u(t;\tau,u_0)$ is the unique solution of (1) with $u(\tau) = u_0$.

We first prove the weak continuity of the process.

Lemma 3.1.1. *Let $\{u_{0_n}\}$ be a sequence in H converging weakly in H to an element $u_0 \in H$. Then*

$$U(t,\tau)u_{0_n} \rightharpoonup U(t,\tau)u_0 \quad \text{weakly in } H, \quad \text{for all } \tau \le t, \tag{15}$$

$$U(t,\tau)u_{0_n} \rightharpoonup U(t,\tau)u_0 \quad \text{weakly in } L^2(\tau,T;V), \quad \text{for all } \tau < T. \tag{16}$$

Proof. Let $u_n(t) = U(t,\tau)u_{0_n}$ and $u(t) = U(t,\tau)u_0$, we have, for all $T > \tau$,

$$\{u_n\} \text{ is bounded in } W = L^\infty(\tau,T;H) \cap L^2(\tau,T;V) \cap L^{s+1}(\tau,T;\mathbb{L}^{s+1}), \tag{17}$$

and

$$\{u_n'\} \text{ is bounded in } W^* = L^{1+\frac{1}{s}}(\tau,T;(\mathbb{W}_0^{2,1+s} \cap V)').$$

Then, for all $v \in V$, and $\tau \le t \le t+a \le T$ with $T > \tau$,

$$(u_n(t+a) - u_n(t), v) = \int_t^{t+a} \langle u_n'(s), v\rangle ds \le \|v\| a^{\frac{1}{2}} \|u_n'\|_{W^*} \le C_T \|v\| a^{\frac{1}{2}}, \tag{18}$$

where C_T is positive, independent of n. Then, for $v = u_n(t+a) - u_n(t)$, which belongs to V for almost every t, we have from (18)

$$|u_n(t+a) - u_n(t)|^2 \leq C_T a^{\frac{1}{2}} \|u_n(t+a) - u_n(t)\|.$$

Hence

$$\int_\tau^{T-a} |u_n(t+a) - u_n(t)|^2 ds \leq C_T a^{\frac{1}{2}} \int_\tau^{T-a} \|u_n(t+a) - u_n(t)\| dt. \tag{19}$$

Using Cauchy-Schwarz's inequality and (17), we have from (19)

$$\int_\tau^{T-a} |u_n(t+a) - u_n(t)|^2 dt \leq \widetilde{C}_T a^{\frac{1}{2}},$$

for another positive constant $\widetilde{C}_T$ independent of n. Therefore

$$\lim_{a\to 0} \sup_n \int_\tau^{T-a} \|u_n(t+a) - u_n(t)\|^2_{\mathbb{L}^2(\Omega_r)} dt = 0, \tag{20}$$

for all $r > 0$, where $\Omega_r = \{x \in \Omega : |x| < r\}$. Moreover, from (17), $\{u_n|_{\Omega_r}\}$ is bounded in $L^\infty(\tau, T; \mathbb{L}^2(\Omega_r)) \cap L^2(\tau, T; \mathbb{H}^1(\Omega_r)) \cap L^{s+1}(\tau, T; \mathbb{L}^{s+1}(\Omega_r))$ for all $r > 0$. Consider now a truncation function $\rho \in C^1(\mathbb{R}^+)$ with $\rho(s) = 1$, in $[0,1]$ and $\rho(s) = 0$,in $[2, +\infty)$. For each $r > 0$, define $v_{n,r}(x) = \rho\left(\frac{|x|^2}{r^2}\right) u_n(x)$ for $x \in \Omega_{2r}$. Then, from (20), we have

$$\lim_{a\to 0} \sup_n \int_\tau^{T-a} \|u_n(t+a) - u_n(t)\|^2_{\mathbb{L}^2(\Omega_{2r})} dt = 0, \quad \forall T > \tau,\ \forall r > 0.$$

Hence $\{v_{n,r}\}$ is bounded in $L^\infty(\tau, T; \mathbb{L}^2(\Omega_{2r})) \cap L^2(\tau, T; \mathbb{H}^1_0(\Omega_{2r})) \cap L^{s+1}(\tau, T; \mathbb{L}^{s+1}(\Omega_{2r}))$, for all $T > \tau$, $r > 0$. Thus, by a compactness theorem with $X = \mathbb{L}^2(\Omega_{2r})$, $Y = \mathbb{H}^1_0(\Omega_{2r})$ and $p = 2$, $\{v_{n,r}\}$ is relatively compact in $L^2(\tau, T; \mathbb{L}^2(\Omega_{2r})$, $\forall T > \tau$, $r > 0$. It follows that $\{u_n|_{\Omega_r}\}$ is relatively compact in $L^2(\tau, T; \mathbb{L}^2(\Omega_{2r})$, $\forall T > \tau$, $r > 0$. Then, by a diagonal process, we can extract a subsequence $\{u_{n'}\}$ such that

$$\begin{aligned} u_{n'} \to \widetilde{u} & \text{ weak-* in } L^\infty_{loc}(\mathbb{R}, H), \\ & \text{weakly in } L^2_{loc}(\mathbb{R}; V) \cap L^{s+1}_{loc}(\mathbb{R}; \mathbb{L}^{s+1}), \\ & \text{strongly in } L^2_{loc}(\mathbb{R}, \mathbb{L}^2(\Omega_r)), \qquad r > 0, \end{aligned} \tag{21}$$

for some

$$\widetilde{u} \in L^\infty_{loc}(\mathbb{R}, H) \cap L^2_{loc}(\mathbb{R}, V) \cap L^{s+1}_{loc}(\mathbb{R}; \mathbb{L}^{s+1}). \tag{22}$$

The convergence (22) allows us to pass to the limit in the equation for $u_{n'}$ to find that $\widetilde{u}$ is a solution of (1) with $\widetilde{u}(\tau) = u_0$. By the uniqueness of the solutions we must have $\widetilde{u} = u$. Then by a contradiction argument we deduce that the whole sequence $\{u_n\}$ converges to u in the sense of (22). This proves (16).

Now, from the strong convergence in (21) we also have that $u_n(t)$ converges strongly in $\mathbb{L}^2(\Omega_r)$ to $u(t)$ for a.e. $t \geq \tau$ and all $r > 0$. Hence for all $v \in \mathcal{V}$,

$$(u_n(t), v) \to (u(t), v) \qquad \text{for a.e. } t \in \mathbb{R}.$$

Moveover, from (19) and (20), we see that $\{(u_n(t), v)\}$ is equibounded and equicontinuous on $[\tau, T]$, for all $T > 0$. Therefore,

$$(u_n(t), v) \to (u(t), v), \qquad \forall t \in \mathbb{R},\ \forall v \in \mathcal{V}.$$

Finally, (15) is followed by the fact that $\mathcal{V}$ is dense in H. □

From now on, we denote

$$\sigma = \nu\lambda_1.$$

Let $\mathcal{R}_\sigma$ be the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t\to-\infty} e^{\sigma t} r^2(t) = 0, \tag{23}$$

and denote by $\mathcal{D}$ the class of all families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(H)$ such that $D(t) \subset B(0, \hat{r}(t))$, for some $\hat{r}(t) \in \mathcal{R}_\sigma$, where $B(0, r)$ denotes the close ball in H, centered at zero with radius r.

Now, we are in a position to prove the main result of the paper.

Theorem 3.2. *Suppose that conditions $(H1)-(H2)$ hold. Then, there exists a unique minimal pullback $\mathcal{D}$-attractor for the process $U(t, \tau)$.*

Proof. Let $\tau \in \mathbb{R}$ and $u_0 \in H$ be fixed, and denote

$$u(t) = u(t; \tau, u_0) = U(t, \tau)u_0, \qquad \text{for all } t \geq \tau.$$

Let $\hat{\mathcal{D}} \in \mathcal{D}$ be given. From (6), we have

$$|U(t, \tau)u_0|^2 \leq e^{-\sigma(t-\tau)}\hat{r}(\tau) + \frac{e^{-\sigma t}}{\nu}\int_{-\infty}^{t} e^{\sigma s}\|f(s)\|_*^2 ds, \tag{24}$$

for all $u_0 \in D(\tau)$, and all $t \geq \tau$.

Denote $R_\sigma(t) \in \mathcal{R}_\sigma$ by

$$R_\sigma^2(t) = \frac{2e^{-\sigma t}}{\nu}\int_{-\infty}^{t} e^{\sigma s}\|f(s)\|_*^2 ds,$$

and consider the family $\hat{\mathcal{B}}_\sigma$ of closed balls in H defined by $B_\sigma(t) = B(0, R_\sigma(t))$. It is straightforward to check that $\hat{\mathcal{B}}_\sigma \in \mathcal{D}$, and moreover, by (23) and (24), the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}$-absorbing for the process U.

According to Theorem 3.1, to finish the proof of the theorem we only have to prove that U is pullback $\mathcal{D}$-asymptotically compact.

Let $\hat{\mathcal{D}} \in \mathcal{D}$, a sequence $\tau_n \to -\infty$, a sequence $u_{n_0} \in D(\tau_n)$ and $t \in \mathbb{R}$, be fixed. We must prove that from the sequence $\{U(t, \tau_n)u_{n_0}\}$ we can extract a subsequence that converges in H.

As the family $\hat{\mathcal{B}}_\sigma$ is pullback $\mathcal{D}$-absorbing, for each integer $k \geq 0$, there exists a $\tau_{\hat{\mathcal{D}}}(k) \leq t - k$ such that

$$U(t-k, \tau)D(\tau) \subset B_\sigma(t-k) \text{ for all } \tau \leq \tau_{\hat{\mathcal{D}}}(k), \tag{25}$$

so that for $\tau_n \leq \tau_{\hat{\mathcal{D}}}(k)$,

$$U(t-k, \tau_n)u_{0_n} \subset B_\sigma(t-k).$$

Thus, $\{U(t-k, \tau_n)u_{0_n}\}$ is weakly precompact in H and since $B_\sigma(t-k)$ is closed and convex, the existence of a subsequence $\{(\tau_{n'}, u_{0_{n'}})\} \subset \{(\tau_n, u_{0_n})\}$, and a sequence $\{w_k : k \geq 0\} \subset H$, such that for all $k \geq 0$, $w_k \in B_\sigma(t-k)$ and

$$U(t-k, \tau_{n'})u_{0_{n'}} \rightharpoonup w_k \text{ weakly in } H, \tag{26}$$

Note then by the weak continuity of $U(t,\tau)$ established in Lemma 3.1.1 that

$$\begin{aligned} w_0 &= \lim_{n'\to\infty H_w} U(t,\tau_{n'})u_{0_{n'}} = \lim_{n'\to\infty H_w} U(t,t-k)U(t-k,\tau_{n'})u_{0_{n'}} \\ &= U(t,t-k)\lim_{n'\to\infty H_w} U(t-k,\tau_{n'})u_{0_{n'}} = U(t,t-k)w_k, \end{aligned}$$

where $\lim_{H_w}$ denotes the limit taken in the weak topology of H. Thus

$$U(t,t-k)w_k = w_0, \qquad \text{for all } k \geq 0. \tag{27}$$

Now, from (26), by the lower semi-continuity of the norm,

$$|w_0| \leq \liminf_{n'\to\infty} |U(t,\tau_{n'})u_{0_{n'}}|.$$

If we now prove that also

$$\limsup_{n'\to\infty} |U(t,\tau_{n'})u_{0_{n'}}| \leq |w_0|, \tag{28}$$

then we will have

$$\lim_{n'\to\infty} |U(t,\tau_{n'})u_{0_{n'}}| = |w_0|,$$

and this, together with the weak convergence, will imply the strong convergence in H of $U(t,\tau_{n'})u_{0_{n'}}$ to w_0.

In order to prove (28), define $[.,.] : V\times V \to \mathbb{R}$ by

$$[u,v] = \nu((u,v)) - \nu\frac{\lambda_1}{2}(u,v), \qquad \forall u,v\in V. \tag{29}$$

Clearly, $[.,.]$ is bilinear and symmetric. Moreover, from the fact that $\|u\|^2 \geq \lambda_1|u|^2$,

$$\begin{aligned} [u]^2 \equiv [u,u] &= \nu\|u\|^2 - \nu\frac{\lambda_1}{2}|u|^2 \\ &\geq \nu\|u\|^2 - \frac{\nu}{2}\|u\|^2 = \frac{\nu}{2}\|u\|^2. \end{aligned}$$

Hence

$$\frac{\nu}{2}\|u\|^2 \leq [u]^2 \leq \nu\|u\|^2, \qquad \forall u\in V. \tag{30}$$

Thus, $[.,.]$ defines an inner product in V with norm $[.] = [.,.]^{\frac{1}{2}}$, which is equivalent to the norm $\|.\|$ in V.

Now, from equation (5), Lemmas 2.0.1 and 2.0.4, we get

$$\frac{d}{dt}|u(t)|^2 + 2\left([u(t)]^2 + \frac{\sigma}{2}|u(t)|^2\right) \leq 2\langle f(t), u(t)\rangle.$$

Multiplying by $e^{\sigma t}$ then integrating from τ to t, we get

$$|u(t)|^2 \leq |u_0|^2 e^{-\sigma(t-\tau)} + 2\int_\tau^t \left(\langle f(s),u(s)\rangle - [u(s)]^2\right) ds.$$

which can be written as

$$|U(t,\tau)u_0|^2 \le |u_0|^2 e^{\sigma(\tau-t)} \\ + 2\int_\tau^t e^{\sigma(s-t)} \left(\langle f(s), U(s,\tau)u_0\rangle - [U(s,\tau)u_0]^2\right) ds, \tag{31}$$

for all $\tau \le t$, and all $u_0 \in H$. Thus, for all $k \ge 0$ and all $\tau_{n'} \le t-k$,

$$\begin{aligned} |U(t,\tau_{n'})u_{0_{n'}}|^2 &= |U(t,t-k)U(t-k,\tau_{n'})u_{0_{n'}}|^2 \\ &\le e^{-\sigma k}|U(t-k,\tau_{n'})u_{0_{n'}}|^2 \\ &\quad + 2\int_{t-k}^t e^{\sigma(s-t)}\langle f(s), U(s,t-k)U(t-k,\tau_{n'})u_{0_{n'}}\rangle ds \\ &\quad - 2\int_{t-k}^t e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0_{n'}}]^2 ds. \end{aligned} \tag{32}$$

As by (25),

$$U(t-k,\tau_{n'})u_{0_{n'}} \in B_\sigma(t-k), \qquad \text{for all } \tau_{n'} \le \tau_{\hat{D}}(k), \quad k \ge 0,$$

we have

$$\limsup_{n'\to\infty} e^{-\sigma k}|U(t-k,\tau_{n'})u_{0_{n'}}|^2 \le e^{-\sigma k}R_\sigma^2(t-k), \quad k \ge 0. \tag{33}$$

As $U(t-k,\tau_{n'})u_{0_{n'}} \rightharpoonup w_k$ weakly in H, from Lemma 3.1.1 we have

$$U(.,t-k)U(t-k,\tau_{n'})u_{0_{n'}} \rightharpoonup U(.,t-k)w_k, \tag{34}$$

weakly in $L^2(t-k,t;V)$.

Taking into account that, in particular, $e^{\sigma(s-t)}f(s) \in L^2(t-k,t;V')$, we obtain from (34),

$$\begin{aligned} \lim_{n'\to\infty} &\int_{t-k}^t e^{\sigma(s-t)}\langle f(s), U(s,t-k)U(t-k,\tau_{n'})u_{0_{n'}}\rangle ds \\ &= \int_{t-k}^t e^{\sigma(s-t)}\langle f(s), U(s,t-k)w_k\rangle ds. \end{aligned} \tag{35}$$

Moreover, since $[.]$ is a norm in V equivalent to $\|.\|$ and

$$0 < e^{-\sigma k} \le e^{\sigma(s-t)} \le 1, \qquad \text{for all } s \in [t-k,t],$$

we see that

$$\left(\int_{t-k}^t e^{-\sigma(t-s)}[.]^2 ds\right)^{\frac{1}{2}}$$

is a norm in $L^2(t-k,t;V)$ equivalent to the usual norm. Hence from (34) we deduce that

$$\begin{aligned} &\int_{t-k}^t e^{\sigma(s-t)}[U(s,t-k)w_k]^2 ds \\ &\quad \le \liminf_{n'\to\infty} \int_{t-k}^t e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0_{n'}}]^2 ds. \end{aligned}$$

Hence

$$\begin{aligned}\limsup_{n'\to\infty} -2\int_{t-k}^{t} & e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0_{n'}}]^2 ds\\ &= -\liminf_{n'\to\infty} 2\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)U(t-k,\tau_{n'})u_{0_{n'}}]^2 ds\\ &\le -2\int_{t-k}^{t} e^{\sigma(s-t)}[U(s,t-k)w_k]^2 ds. \end{aligned}\tag{36}$$

We can now pass to the lim sup as n' goes to infinity in (32), taking (33), (35) and (36) into account to obtain

$$\begin{aligned}\limsup_{n'\to\infty} & |U(t,\tau_{n'})u_{0_{n'}}|^2\\ &\le e^{-\sigma k}R_\sigma^2(t-k)\\ &\quad + 2\int_{t-k}^{t} e^{\sigma(s-t)}\left(\langle f(s),U(s,t-k)w_k\rangle - [U(s,t-k)w_k]^2\right) ds. \end{aligned}\tag{37}$$

On the other hand, we obtain from (31) applied to (27) that

$$\begin{aligned}|w_0| &= |U(t,t-k)w_k|^2\\ &= |w_k|^2 e^{-\sigma k}\\ &\quad + 2\int_{t-k}^{t} e^{\sigma(s-t)}\left(\langle f(s),U(s,t-k)w_k\rangle - [U(s,t-k)w_k]^2\right) ds. \end{aligned}\tag{38}$$

From (37) and (38), we have

$$\begin{aligned}\limsup_{n'\to\infty}|U(t,\tau_{n'})u_{0_{n'}}|^2 &\le e^{-\sigma k}R_\sigma^2(t-k)+|w_0|^2-|w_k|^2e^{-\sigma k}\\ &\le e^{-\sigma k}R_\sigma^2(t-k)+|w_0|^2,\end{aligned}$$

and thus, taking into account that

$$e^{-\sigma k}R_\sigma^2(t-k) = \frac{2e^{-\sigma t}}{\nu}\int_{-\infty}^{t-k} e^{\sigma s}\|f(s)\|_*^2 ds \to 0,$$

when $k\to+\infty$, we easily obtain (28) from the last inequality. □

Remark 3.1. *Because of the tempered condition in the definition of $\mathcal{D}$, as a corollary of the results in the paper [12], one may establish the existence of a global pullback attractor for a different universe, that of fixed bounded set of H; this attractor is a subset of the attractor obtained in Theorem 3.2.*

4. Stability of stationary solutions

Let us consider the equation

$$\frac{du}{dt} + A\varphi(u) + B(u,u) = f,\tag{39}$$

with $f\in V'$ independent of t. A stationary solution to (39) is $u\in V\cap\mathbb{L}^{s+1}$ such that

$$\langle A\varphi(u)+B(u,u),v\rangle = \langle f,v\rangle,\quad \forall t\ge 0,\ \forall v\in V.\tag{40}$$

In this section, we assume that

$$s \geq \begin{cases} N+1, & \text{if } N \geq 3 \\ 2, & \text{if } N = 2. \end{cases} \tag{41}$$

Lemma 4.0.1. *Assume $u(t)$ be the unique solution of (5) with $\tau = 0$ and $f(t) \equiv f$, for all $t \geq 0$ and s as in (41), there exists $\rho_u > 0$ and $\delta > 0$ such that*

$$\|u\|^2_{L^\infty(\mathbb{R}^+;H)} \leq \rho_u, \tag{42}$$

$$\int_0^t \|u(y)\|^{s+1}_{\mathbb{L}^{s+1}} dy \leq \left(\alpha - \frac{C_1 \rho_u}{\lambda_1 \delta^s}\right)^{-1} \left[\frac{1}{2}|u_0|^2_* + \left(\alpha \xi_0^{s+1}|\Omega| + \delta\rho_u + \|f\|_* \sqrt{\frac{\rho_u}{\lambda_1}}\right) t\right]. \tag{43}$$

Proof. Multiplying (5) by $u(t)$ and integrating over Ω, we have

$$\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \leq \frac{1}{\nu}\|f\|^2_*.$$

Noting that $\|u\|^2 \geq \lambda_1 |u|^2$, we have

$$\frac{d}{dt}|u(t)|^2 + \nu\lambda_1|u(t)|^2 \leq \frac{1}{\nu}\|f\|^2_*.$$

We have

$$\frac{d}{dt}(e^{\nu\lambda_1 t}|u(t)|^2) \leq \frac{\|f\|^2_*}{\nu} e^{\nu\lambda_1 t}.$$

Integrating from 0 to t, we get

$$|u(t)|^2 \leq e^{-\nu\lambda_1 t}|u_0|^2 + \frac{\|f\|^2_*}{\nu^2\lambda_1}(1 - e^{-\nu\lambda_1 t}). \tag{44}$$

We obtain the estimate

$$\|u\|^2_{L^\infty(\mathbb{R}^+;H)} \leq \rho_u = \max\left(|u_0|^2; \frac{\|f\|^2_*}{\lambda_1 \nu^2}\right).$$

Arguing as we did for the inequality (9), we obtain

$$\begin{aligned} \alpha \int_0^t \|u(y)\|^{s+1}_{\mathbb{L}^{s+1}} dy &\leq \frac{1}{2}|u_0|^2_* + \frac{1}{\sqrt{\lambda_1}} \int_0^t \|f\|_* |u(y)| dy + \int_0^t \alpha\xi_0^{s+1}|\Omega| dy \\ &+ \int_0^t \left(\delta|u(y)|^2 + \frac{C_1}{\lambda_1\delta^s}\|u(y)\|^{s+1}_{\mathbb{L}^{s+1}}|u(y)|^2\right) dy \\ &\leq \frac{1}{2}|u_0|^2_* + \left(\alpha\xi_0^{s+1}|\Omega| + \delta\rho_u + \|f\|_*\sqrt{\frac{\rho_u}{\lambda_1}}\right) t + \frac{C_1\rho_u}{\lambda_1\delta^s}\int_0^t \|u(y)\|^{s+1}_{\mathbb{L}^{s+1}} dy, \end{aligned}$$

hence, the result. □

From (44), we immediately infer

$$\limsup_{t\to+\infty} |u(t)| \leq \frac{\|f\|_*}{\nu\sqrt{\lambda_1}} = \rho. \tag{45}$$

Note that $\rho_u = \rho^2$ if $|u_0| \leq \rho$.

4.1. *Existence and uniqueness of stationary solutions*

Theorem 4.1. *Suppose that φ satisfies conditions $(H1)$ with s as in (41). Then we have the following*

(1) For all $f \in V'$, there exists a stationary solution to (39).

(2) If u be a solution of (39), let ρ as in (45), and let $\delta > \left(\frac{C_1\rho^2}{\alpha\lambda_1}\right)^{\frac{1}{s}}$. Then,

$$|u| \leq \rho, \qquad \|u\| \leq \sqrt{\lambda_1}\rho$$
$$\|u\|^{s+1}_{\mathbb{L}^{s+1}} \leq \left(\alpha - \frac{C_1\rho^2}{\lambda_1\delta^s}\right)^{-1}\left(\alpha\xi_0^{s+1}|\Omega| + \delta\rho^2 + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right).$$

(3) There exists $C_1, C_2 > 0$ such that, if

$$\alpha\lambda_1\nu^{s+1} > 2^{s+3}C_2\left[\alpha\xi_0^{s+1}|\Omega| + \frac{(2C_1)^{\frac{1}{s}}\rho^{\frac{2s+2}{s}}}{(\alpha\lambda_1)^{\frac{1}{s}}} + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right] \tag{46}$$

then the stationary solution to (39) is unique.

Proof. (1) Consider $w_1, \cdots, w_m, \cdots$ of elements of $V \cap \mathbb{L}^{s+1}$ which is the orthonormal of H. For each fixed integer $m \geq 1$, we would like to define an approximate solution of (39) by

$$u^m = \sum_{i=1}^{m}\xi_{i,m}w_i, \quad \xi_{i,m} \in \mathbb{R}, \tag{47}$$
$$\langle A\varphi(u^m), w_k\rangle + b(u^m, u^m, w_k) = \langle f, w_k\rangle, \quad k = 1, \cdots, m. \tag{48}$$

The equations (47)-(48) are a system of nonlinear equations for $\xi_{1,m}, \cdots, \xi_{m,m}$, and the existence of a solution of this system is not obvious. Consider X be the space spanned by $w_1, \cdots, w_m$; the scalar product on X is the scalar product $((.))$ induced by V, and $P = P_m$ is defined by

$$[P_m(u), v] = ((P_m(u), v)) = \langle A\varphi(u), v\rangle + b(u, u, v) - \langle f, v\rangle, \qquad \forall u, v \in X.$$

The continuity of the mapping P_m is obvious.

$$\begin{aligned}[P_m(u), u] &= \langle A\varphi(u), u\rangle + b(u, u, u) - \langle f, u\rangle \\ &\geq \nu\|u\|^2 - \langle f, u\rangle \\ &\geq \nu\|u\|^2 - \|f\|_*\|u\|,\end{aligned}$$

$$[P_m(u), u] \geq \|u\|(\nu\|u\| - \|f\|_*).$$

It follows that $[P_m(u), u] > 0$ for $\|u\| = k$, and k large enough; more precisely, $k > \frac{\|f\|_*}{\nu}$. Using Lemma 1.4 in [15, page 152] we get a solution u^m of (47)-(48).

We multiply (48) by $\xi_{k,m}$ and add the corresponding equalities for $k = 1, \cdots, m$; this gives

$$\langle A\varphi(u^m), u^m\rangle + b(u^m, u^m, u^m) = \langle f, u^m\rangle$$

or, using Lemmas 2.0.1 and 2.0.4,

$$\nu\|u^m\|^2 \leq \langle f, u^m\rangle \leq \|f\|_*\|u^m\|.$$

We obtain the a priori estimate:

$$\|u^m\| \leq \frac{\|f\|_*}{\nu} = C_{10}. \tag{49}$$

Since $V \hookrightarrow H$, we have

$$|u^m| \leq \frac{1}{\sqrt{\lambda_1}}\|u^m\| \leq \frac{C_{10}}{\sqrt{\lambda_1}}. \tag{50}$$

We get a constant (independent of m) C_{11} such that

$$\|u^m\|_{\mathbb{L}^{s+1}}^{s+1} \leq C_{11}. \tag{51}$$

Indeed, using Lemma 2.0.5, we get

$$(\varphi(u^m), u^m) \geq \alpha\|u^m\|_{\mathbb{L}^{s+1}}^{s+1} - \alpha\xi_0^{s+1}|\Omega|.$$

Moreover, using Lemma 2.0.3 with $\mu = \delta$, we get

$$|b(u^m, u^m, A^{-1}u^m)| \leq \delta|u^m|^2 + \frac{C_1}{\lambda_1\delta^s}\|u^m\|_{\mathbb{L}^{s+1}}^{s+1}|u^m|^2.$$

From equation (48), we have

$$(\varphi(u^m), u^m) + b(u^m, u^m, A^{-1}u^m) = \langle f, A^{-1}u^m\rangle \leq \frac{\|f\|_*}{\sqrt{\lambda_1}}|u^m|.$$

Hence,

$$\alpha\|u^m\|_{\mathbb{L}^{s+1}}^{s+1} \leq \alpha\xi_0^{s+1}|\Omega| + \delta|u^m|^2 + \frac{C_1}{\lambda_1\delta^s}\|u^m\|_{\mathbb{L}^{s+1}}^{s+1}|u^m|^2 + \frac{\|f\|_*}{\sqrt{\lambda_1}}|u^m|.$$

Take account (50) into, we get

$$\|u^m\|_{\mathbb{L}^{s+1}}^{s+1} \leq \left(\alpha - \frac{C_1C_{10}^2}{\lambda_1^2\delta^s}\right)^{-1}\left(\alpha\xi_0^{s+1}|\Omega| + \delta\frac{C_{10}^2}{\lambda_1} + \frac{\|f\|_*C_{10}}{\lambda_1}\right).$$

Therefore, (51) follow from the fact that we can choose δ such that $\alpha - \dfrac{C_1C_{10}^2}{\lambda_1^2\delta^s} > 0$.

By condition (2), $\sigma \in C^1$ implies σ is bounded on $[0, \xi_0]$, i.e., $\sigma(\xi) \leq C_\sigma$ for all $\xi \in [0, \xi_0]$. Thus,

$$\begin{aligned}
\|\varphi(u)\|_{\mathbb{L}^{1+\frac{1}{s}}}^{1+\frac{1}{s}} &= \int_{\Omega_1}|\varphi(u(x))|^{1+\frac{1}{s}}dx + \int_{\Omega_2}|\varphi(u(x))|^{1+\frac{1}{s}}dx \\
&\leq \xi_0^{1+\frac{1}{s}}\int_{\Omega_1}|\sigma(|u(x)|)|^{1+\frac{1}{s}}dx + \int_{\Omega_2}\left(\beta|u(x)|^s\right)^{1+\frac{1}{s}}dx \\
&\leq \xi_0^{1+\frac{1}{s}}\int_{\Omega_1}C_\sigma^{1+\frac{1}{s}}dx + \beta^{1+\frac{1}{s}}\int_{\Omega_2}|u(x)|^{1+s}dx \\
&\leq \xi_0^{1+\frac{1}{s}}C_\sigma^{1+\frac{1}{s}}|\Omega| + \beta^{1+\frac{1}{s}}\|u\|_{\mathbb{L}^{1+s}}^{1+s},
\end{aligned}$$

where

$$\Omega_1 = \{x \in \Omega : |u(x)| < \xi_0\} \text{ and } \Omega_2 = \{x \in \Omega : |u(x)| \geq \xi_0\}.$$

So, we get

$$\|\varphi(u^m)\|_{\mathbb{L}^{1+\frac{1}{s}}}^{1+\frac{1}{s}} \leq \xi_0^{1+\frac{1}{s}}C_\sigma^{1+\frac{1}{s}}|\Omega| + \beta^{1+\frac{1}{s}}C_{11} = C_{13}. \tag{52}$$

Finally, using Lemma 2.0.6 with $r = s+1 \geq N$, we have

$$\|B(u^m, u^m)\|_* \leq C_4 \|u^m\|^{\frac{N}{s+1}} |u^m|^{\frac{s+1-N}{s+1}} \|u^m\|_{\mathbb{L}^{s+1}} \leq C_4 \lambda_1^{-\frac{s+1-N}{2(s+1)}} C_{10} C_{11}^{\frac{1}{s+1}} = C_{14}.$$

Therefore, we have

$$\begin{array}{ll} u^m \to u, & \text{weakly in } V, \\ u^m \to u, & \text{weakly in } \mathbb{L}^{s+1}, \\ \varphi(u^m) \to \chi, & \text{weakly in } \mathbb{L}^{1+\frac{1}{s}}, \\ B(u^m, u^m) \to \psi, & \text{weakly in } V', \end{array}$$

up to a subsequence.

Notice that

$$V \subset\subset H \subset (\mathbb{W}_0^{2,s+1} \cap V)'$$

is an evolution triplet, applying the Compactness Lemma [10] we can assume that $u^m \to u$ strongly in H. Hence $u^m \to u$ a.e. in $\Omega \times [\tau, T]$. Since φ is continuous, it follows that $\varphi(u^m) \to \varphi(u)$ a.e. in $\Omega \times [\tau, T]$. Since weak limit is unique and thanks to Lemma 1.3 in [10, Chapter 1], one has

$$\varphi(u^m) \to \varphi(u), \text{ in } \mathbb{L}^{1+\frac{1}{s}}.$$

Similarly,

$$B(u^m, u^m) \to B(u, u), \text{ in } V'.$$

Now, we can pass to the limit in (48) with the subsequence, we find that

$$\langle A\varphi(u), v\rangle + b(u, u, v) = \langle f, v\rangle \tag{53}$$

for any $v = w_1, \cdots, w_m, \cdots$. Equation (53) is also true for any v which is linear combination of $w_1, \cdots, w_m, \cdots$. Since these combinations are dense in V, continuity argument shows finally that (53) holds for each $v \in V$ and that is a solution of equation (39).

(2) Arguing as we did for the inequalities (49), (50) and (51), we have the result.

(3) Now let u_1 and u_2 be two different solutions of (46) and let $u = u_1 - u_2$. We substract the equation (46) corresponding to u_1 and u_2 and obtain

$$\langle A\varphi(u_1) - A\varphi(u_2), v\rangle + b(u_1, u, v) + b(u, u_2, v) = 0, \quad \forall v \in V. \tag{54}$$

We take $v = A^{-1}u$ in (54) and using Lemma 2.0.5 with $\mu = \dfrac{\nu}{2}$, we get

$$\nu|u|^2 \leq \frac{\nu}{2}|u|^2 + C_2\left(\frac{\nu}{2}\right)^{-s}\left(\|u_1\|_{\mathbb{L}^{s+1}}^{s+1} + \|u_2\|_{\mathbb{L}^{s+1}}^{s+1}\right)\|u\|_*^2,$$

whence

$$\begin{aligned} \nu|u|^2 &\leq \frac{2C_2}{\lambda_1}\left(\frac{\nu}{2}\right)^{-s}\left(\|u_1\|_{\mathbb{L}^{s+1}}^{s+1} + \|u_2\|_{\mathbb{L}^{s+1}}^{s+1}\right)|u|^2 \\ &\leq \frac{4C_2}{\lambda_1}\left(\frac{\nu}{2}\right)^{-s}\left(\alpha - \frac{C_1\rho^2}{\lambda_1\delta^s}\right)^{-1}\left(\alpha\xi_0^{s+1}|\Omega| + \delta\rho^2 + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right)|u|^2. \end{aligned}$$

Choose $\delta = \left(\dfrac{2C_1\rho^2}{\alpha\lambda_1}\right)^{\frac{1}{s}}$, we get

$$\nu|u|^2 \leq \frac{2^{s+2}C_2}{\lambda_1\nu^s}\frac{2}{\alpha}\left[\alpha\xi_0^{s+1}|\Omega| + \frac{(2C_1)^{\frac{1}{s}}\rho^{\frac{2s+2}{s}}}{(\alpha\lambda_1)^{\frac{1}{s}}} + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right]|u|^2.$$

Because (46) this inequality implies $|u| = 0$, which means $u_1 = u_2$. □

Remark 4.1. *Temam [15, Theorem 1.3, page 145] has proved that if $\nu^2 > C\|f\|_*$ then problem*

$$-\nu Au + B(u,u) = f$$

(39) *have a unique solution.*

Noting that, this is our problem in the case $N = 2$, $s = 1$, $\alpha = \nu$, $\xi_0 = 0$. Then (46) can rewrite that

$$\nu^3 > \frac{2^{1+3}C_2}{\lambda_1}\left[\frac{2C_1}{\nu\lambda_1}\frac{\|f\|_*^4}{\nu^4\lambda_1^2} + \frac{\|f\|_*^2}{\lambda_1\nu}\right] \geq C^3\frac{\|f\|_*^3}{\nu^3}.$$

This implies that

$$\nu^2 > C\|f\|_*.$$

4.2. *Exponential convergence of solutions*

Lemma 4.1.1. *Assume that $|u_0| \leq 2\rho$ (ρ as in (45)). If ν is sufficiently large or f sufficiently small so that*

$$\alpha\lambda_1\nu^{s+1} > 2^{s+2}C_2\left[\alpha\xi_0^{s+1}|\Omega| + \frac{(2C_1)^{\frac{1}{s}}\rho^{\frac{2s+2}{s}}}{(\alpha\lambda_1)^{\frac{1}{s}}} + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right]. \tag{55}$$

Then there is a unique stationary solution u_∞ of (39) and every solution $u(t)$ of problem (5) (with initial data u_0 and $f(t) \equiv f$) converges to u_∞ exponentially fast as $t \to +\infty$. More precisely, we have

$$\lim_{t\to+\infty}\|u(t) - u_\infty\|_* = 0.$$

Proof. We set $w(t) = u(t) - u_\infty$, and observe that

$$\frac{d}{dt}w(t) + A\varphi(u(t)) - A\varphi(u_\infty) + B(u(t),u(t)) - B(u_\infty,u_\infty) = 0.$$

Take inner product by $A^{-1}w(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_*^2 + (\varphi(u(t)) - \varphi(u_\infty), w(t)) \leq |b(u(t),w(t),A^{-1}w(t)) + b(w(t),u_\infty,A^{-1}w(t))|.$$

Using Lemma 2.0.4, we have

$$2(\varphi(u(t)) - \varphi(u_\infty), w(t)) \geq 2\nu|w(t)|^2 \geq 2\nu\lambda_1\|w(t)\|_*^2.$$

Using Lemma 2.0.5, we have

$$\begin{aligned}|b(u(t),w(t),A^{-1}w(t)) + b(w(t),u_\infty,A^{-1}w(t))|\\ \leq \frac{\nu}{2}|w(t)|^2 + C_2\left(\frac{\nu}{2}\right)^{-s}\left(\|u(t)\|_{\mathbb{L}^{s+1}}^{s+1} + \|u_\infty\|_{\mathbb{L}^{s+1}}^{s+1}\right)\|w(t)\|_*^2.\end{aligned}$$

Hence,

$$\frac{\frac{d}{dt}\|w(t)\|_*^2}{\|w(t)\|_*^2} \leq C_2 2^s\nu^{-s}\left(\|u(t)\|_{\mathbb{L}^{s+1}}^{s+1} + \|u_\infty\|_{\mathbb{L}^{s+1}}^{s+1}\right) - \nu\lambda_1. \tag{56}$$

Integrating (56) from 0 to t, and using Lemma 4.0.1, Theorem 4.1, we obtain

$$\log \frac{\|w(t)\|_*^2}{\|w_0\|_*^2} \leq C_{15}t + C_{16}.$$

Therefore

$$\|w(t)\|_*^2 \leq e^{C_{16}}\|w_0\|_*^2 e^{C_{15}t}, \tag{57}$$

where

$$C_{16} = C_2 2^{s-1}\nu^{-s}\|u_0\|_*^2 \left(\alpha - \frac{C_1\rho^2}{\lambda_1\delta^s}\right)$$
$$C_{15} = \frac{2^{s+1}C_2}{\nu^s}\left(\alpha - \frac{C_1\rho^2}{\lambda_1\delta^s}\right)^{-1}\left(\alpha\xi_0^{s+1}|\Omega| + \delta\rho^2 + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right) - \nu\lambda_1.$$

Choose $\delta = \left(\frac{2C_1\rho^2}{\alpha\lambda_1}\right)^{\frac{1}{s}}$, we get

$$C_{15} \leq \frac{2^{s+1}C_2}{\nu^s}\frac{2}{\alpha}\left[\alpha\xi_0^{s+1}|\Omega| + \frac{(2C_1)^{\frac{1}{s}}\rho^{\frac{2s+2}{s}}}{(\alpha\lambda_1)^{\frac{1}{s}}} + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right] - \nu\lambda_1.$$

Because (55), we have $C_{15} < 0$, and letting $t \to +\infty$ in (58) we have

$$\lim_{t\to+\infty}\|u(t) - u_\infty\|_* = 0.$$

□

Theorem 4.2. *Assume that $u_0 \in H$, ρ as in (45). If ν is sufficiently large or f sufficiently small so that*

$$\alpha\lambda_1\nu^{s+1} > 2^{s+2}C_2\left[\alpha\xi_0^{s+1}|\Omega| + \frac{(2C_1)^{\frac{1}{s}}\rho^{\frac{2s+2}{s}}}{(\alpha\lambda_1)^{\frac{1}{s}}} + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right]. \tag{58}$$

Then there is a unique stationary solution u_∞ of (39) and every solution $u(t)$ of problem (5) (with initial data u_0 and $f(t) \equiv f$) converges to u_∞ exponentially fast as $t \to +\infty$. More precisely, we have

$$\lim_{t\to+\infty}\|u(t) - u_\infty\|_* = 0.$$

Proof. From (45) we infer that there exists $\bar{t} > 0$ such that $|u(t)| \leq 2\rho$ for all $t \geq \bar{t}$. Now, consider $v_0 = u(\bar{t})$ and let $v(t)$ be the unique solution of (5) with initial data v_0. Therefore

$$\lim_{t\to+\infty}\|u(t) - u_\infty\|_* = \lim_{t\to+\infty}\|v(t) - u_\infty\|_* = 0$$

and the result follows. □

Remark 4.2. *In the case, $N = 2$, $s = 1$, proceding as in Theorem 1.3 of [15], the results of Theorems 4.1 and 4.2 still hold.*

Remark 4.3. *This result covers Gazzola's result in [4] for the case $\xi_0 = 0$, $N = 3$, $s = 4$. Indeed, the condition (58) in the special case $C_1 = 512$ and $C_2 = 4$ becomes*

$$\lambda_1\nu > \frac{256}{\nu^4}\frac{2}{\alpha}\left[\frac{\sqrt{32\rho}}{\sqrt[4]{\alpha\lambda_1}}\rho^2 + \frac{\|f\|_*}{\sqrt{\lambda_1}}.\rho\right].$$

Clearly, this inequality is better than Gazzola's in [4, Theorem 6.2]. He required that $c_8 < 0$, *i.e.,*

$$\nu\lambda_1 > \frac{256}{\nu^4}\left[\left(\frac{\theta\|f\|_*}{\nu}\right)^5 + \frac{2}{\alpha}\left(\frac{\sqrt{32\rho}}{\sqrt[4]{\alpha\lambda_1}}\rho^2 + \frac{2\|f\|_*\rho}{\sqrt{\lambda_1}}\right)\right],$$

where $\theta = \theta(\Omega)$ *is the constant of the embedding* $V \hookrightarrow \mathbb{L}^5$.

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